

QUANTUM NOISE AND SQUEEZING IN OPTICAL PARAMETRIC OSCILLATOR WITH ARBITRARY OUTPUT COUPLING

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Abstract

The redistribution of intrinsic quantum noise in the quadratures of the field generated in a sub-threshold degenerate optical parametric oscillator exhibits interesting dependences on the individual output mirror transmittances, when they are included exactly. We present here a physical picture of this problem, based on mirror boundary conditions, which is valid for arbitrary transmittances and so applies uniformly to all values of the cavity Q factor representing in the opposite extremes perfect oscillator and amplifier configurations. Beginning with a classical second-harmonic pump, we shall generalize our analysis to apply to finite amplitude and phase fluctuations of the pump.

1 Introduction

A degenerate optical parametric oscillator (DOPO) has long been considered a nearly ideal squeezing device when operated just below threshold. The quantum fluctuations of the generated sub-harmonic field are rather immune to spontaneous emission since the two-photon transition governing the parametric down-conversion process sees no resonant intermediate levels.

Nearly all prior work dealing with this problem [1,2,3] has been limited to the situation in which the DOPO cavity is nearly perfect. In a general approach [4,5] developed recently by the author and Abbott, which is based on the exact treatment of mirror boundary conditions, it has become possible to discuss cavity problems in quantum optics for the entire range of cavity transmissions possible. In the present DOPO context, this approach thus permits the extreme limits of a single-pass amplifier (cavity transmission $\rightarrow 100\%$) and of a nearly perfect DOPO cavity (cavity transmission $\rightarrow 0\%$), and all intermediate- Q oscillator configurations to be treated on the same footing. By employing this viewpoint (which may be viewed as a generalization of Collett and Gardiner's approach [2]), we also develop a physically insightful picture of the general squeezing

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problem, one which emphasizes the correlations of the input, output, and intracavity fields that govern the relationship of intracavity and output field fluctuations. Any reference to modes is altogether avoided here.

After treating the DOPO problem with a perfectly monochromatic pump, we shall model realistic experiments in which the pump field has finite amplitude and phase fluctuations. Although any amplitude noise of the pump has a relatively minor impact on the squeezing of the subharmonic signal field, pump phase diffusion even when it is tracked can cause a severe degradation of that squeezing. More detailed discussions of this problem will appear elsewhere [6].

2 Mathematical Formalism

A description of the problem at hand that covers the whole gamut of cavity transmission factors is necessarily multimode in character. We avoid all reference to cavity modes by writing the fully quantized signal field inside the cavity in terms of its rightward (positive- z) and leftward (negative- z) propagating parts. For the positive-frequency part, this decomposition is written in the Heisenberg picture (HP) as

$$E^{(+)}(z, t) = (e_+(z, t) e^{ik_0 z} + e_-(z, t) e^{-ik_0 z}) e^{-i\Omega_0 t}, \quad (1)$$

in which the operators $e_{\pm}(z, t)$ have expectation values that are assumed slowly varying in space and time on the scale of the central wavelength $2\pi/k_0$ and period $2\pi/\Omega_0$.

The parametric interaction of $E^{(+)}(z, t)$ with an intense quasimonochromatic is described via the interaction Hamiltonian (also written in HP) in a cavity of length ℓ filled with the parametric medium:

$$H'_{DOPO} = \frac{3A}{4} \chi^{(2)} e_{pump}^* \int_0^{\ell} [e_+^2(z, t) + e_-^2(z, t)] dz + \text{Hermitian Conjugate} \quad (2)$$

The complex pump amplitude e_{pump} is at most slowly varying in time. The constants A and $\chi^{(2)}$ are the cross-sectional area of the cavity and nonlinear susceptibility, respectively. The notation used is the same as in Ref. [7]. We may write the equations of propagation for $e_{\pm}(z, t)$ in the slowly-varying envelope approximation as

$$\left(\frac{\partial}{\partial z} \pm \frac{1}{c} \frac{\partial}{\partial t} \right) e_{\pm}(z, t) = \pm \frac{ik}{2\epsilon} p_{\pm}^{NL}(z, t), \quad (3)$$

in which the nonlinear polarization waves $p_{\pm}^{NL}(z, t)$ driving the parametric interaction are given by a functional differentiation of the quadratic interaction Hamiltonian (2):

$$\begin{aligned} p_{\pm}^{NL}(z, t) &= -\frac{1}{A} (\delta / \delta e_{\pm}^{\dagger}(z, t)) H'_{DOPO} \\ &= -\frac{3}{4} \chi^{(2)*} e_{pump} e_{\pm}^{\dagger}(z, t). \end{aligned} \quad (4)$$

Thus, on combining (3) and (4), we have the following generalization of the single-mode equations describing the parametric amplification process:

$$\left(\frac{\partial}{\partial z} \pm \frac{1}{c} \frac{\partial}{\partial t} \right) e_{\pm}(z, t) = \pm q e_{\pm}^{\dagger}(z, t), \quad (5)$$



Fig. 1. The DOPO Cavity with End Mirrors at $z = 0$ and $z = \ell$.

To complete the formalism, we supplement Eq. (5) with boundary connections of the *intra-cavity* $e_{\pm}(z, t)$ fields with the *input* vacuum fields. These connections are

$$\begin{aligned} e_+(0, t) &= -\bar{r}e_-(0, t) + \bar{t}e_+^{vac}(0, t); \\ e_-(\ell, t) &= -\bar{r}'e_+(\ell, t) + \bar{t}'e_-^{vac}(\ell, t), \end{aligned} \quad (6)$$

in which e_{\pm}^{vac} are the two traveling pieces of the vacuum field entering the cavity through its mirrors at $z = 0$ and $z = \ell$ with inside-to-outside reflection and transmission coefficients $(-\bar{r}, \bar{t})$ and $(-\bar{r}', \bar{t}')$ respectively (see Fig. 1).

3 The Parametric Amplifier Problem

Without the cavity mirrors, the oscillator reduces to the amplifier configuration in which the two traveling parts e_+ and e_- are not coupled to each other. We may therefore concentrate on only one of them, say the e_+ -field.

Furthermore, for simplicity, we shall assume in this section that the pump has no amplitude and phase randomness, so that it is strictly monochromatic. For this case, one may assume without any loss of generality that q is real and positive, for any constant nonzero phase ϕ_q of q may be scaled out by redefining $e_+(z, t)$ to carry a constant phase factor $\exp(i\phi_q/2)$:

$$e_+(z, t) \rightarrow e_+(z, t) e^{i\phi_q/2}, \quad (7)$$

without altering the physics.

By adding to Eq. (5) and by subtracting from it its Hermitian conjugate, one obtains the

following pair of uncoupled equations for the quadratures of e_+ :

$$\left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t}\right) X_+(z, t) = q X_+(z, t); \quad \left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t}\right) Y_+(z, t) = -q Y_+(z, t), \quad (8)$$

where $X_+(z, t) = \frac{1}{2} (e_+(z, t) + e_+^\dagger(z, t))$; $Y_+(z, t) = \frac{1}{2i} (e_+(z, t) - e_+^\dagger(z, t))$ are the in-phase and $\pi/2$ out-of-phase quadratures. The solution of Eqs. (8) is straightforward in terms of the retarded time variable, $\tau = t - z/c$:

$$X_+(z, t) = X_+(0, t - z/c) e^{qz}; \quad Y_+(z, t) = Y_+(0, t - z/c) e^{-qz}; \quad (9)$$

which represents a phase-sensitive amplification process characteristic of the parametric interaction. These solutions are entirely equivalent to the following time-evolution equations

$$X_+(z, t) = X_+(z - ct, 0) e^{qct}; \quad Y_+(z, t) = Y_+(z - ct, 0) e^{-qct}. \quad (10)$$

The linear relationships of Eqs. (9) or (10) indicate that both the expectation value and fluctuations about it of the X_+ -quadrature (Y_+ -quadrature) of the signal field amplify (attenuate) by the same factor. This statement, valid both classically and quantum-mechanically, clearly implies that any noise initially present in the signal is stretched along the X -quadrature and shrinks along the Y -quadrature, as shown in Fig. 2. It is in this way that quadrature squeezing comes about in a parametric amplifier.

4 The Parametric Oscillator Problem

Our treatment of the parametric oscillator builds upon the simple amplifier analysis presented in Sec. 3 by limiting z to lie between 0 and ℓ and adding mirrors at $z = 0$ and at $z = \ell$, which serve to connect e_+ and e_- and the input vacuum fields via (6). As in Sec. 3, we restrict our analysis here to a perfectly monochromatic pump wave for which Eqs. (9) describe the interaction of the e_+ wave with the medium. Similar relations may be written down for the quadratures of the e_- -field (integrated backwards from $z = \ell$):

$$\begin{aligned} X_-(z, t) &= X_-(\ell, t - (\ell - z)/c) e^{q(\ell - z)}; \\ Y_-(z, t) &= Y_-(\ell, t - (\ell - z)/c) e^{-q(\ell - z)} \end{aligned} \quad (11)$$

Since we are ultimately interested in calculating the quadrature squeezing of the intracavity field $e_\pm(z, t)$, we concentrate here onwards on the quantum fluctuations alone of the various quadratures. We first consider what the implications of the boundary connection relations (6) are for the fluctuations. Since (\tilde{r}, \tilde{t}) and (\tilde{r}', \tilde{t}') are all real, these relations are formally the same as those obeyed by any particular quadrature of e_\pm and e_\pm^{vac} fields, including their X - and Y -quadratures separately. Furthermore, the two fields (or their quadratures) on the right-hand side (RHS) of each equation in (6) are uncorrelated at any t . To see this, we note, for example, that the $e_+^{\text{vac}}(0, t)$ field entering the $z = 0$ mirror contributes to the $e_-^{\text{vac}}(\ell, t')$ field only after a time $t' - t = 2\ell/c$ during which the former field makes a full round trip through the cavity. Thus, $e_-(0, t)$ is correlated with $e_+^{\text{vac}}(0, t - 2\ell/c)$ which is not correlated with $e_+^{\text{vac}}(0, t)$, since the vacuum field fluctuations are essentially δ -correlated in time. In view of this lack of correlation, we may

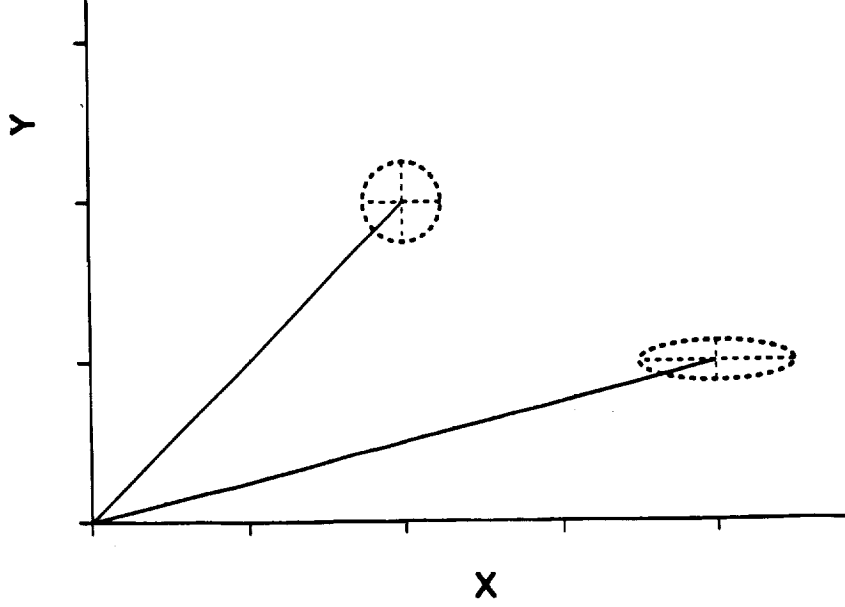


Fig. 2. The Parametric Amplification Process. The X -quadrature is amplified by a given factor (taken to be 2 here) while the Y -quadrature is attenuated by the same factor.

write for the quantum-mechanical variance of, say, the Y -quadrature of fields at the mirrors in terms of the power reflection and transmission coefficients (R, T) and (R', T') (with $R = \tilde{r}^2$, etc.)

$$\begin{aligned} \langle \Delta Y_+(0, t)^2 \rangle &= R \langle \Delta Y_-(0, t)^2 \rangle + T \langle \Delta Y_+^{vac}(0, t)^2 \rangle; \\ \langle \Delta Y_-(\ell, t)^2 \rangle &= R' \langle \Delta Y_+(\ell, t)^2 \rangle + T' \langle \Delta Y_-^{vac}(\ell, t)^2 \rangle, \end{aligned} \quad (12)$$

while setting $z = \ell$ in Eqs. (9) and $z = 0$ in Eqs. (11) yields for the propagation of variances through the medium

$$\langle \Delta Y_+(\ell, t)^2 \rangle = \langle \Delta Y_+(0, t - \ell/c)^2 \rangle e^{-2q\ell}; \quad \langle \Delta Y_-(0, t)^2 \rangle = \langle \Delta Y_-(\ell, t - \ell/c)^2 \rangle e^{-2q\ell}. \quad (13)$$

With the aid of Eqs. (12) and (13), we may express the retarded propagation of the Y_+ -variance at $z = 0$ in one complete round trip as

$$\begin{aligned} \langle \Delta Y_+(0, t)^2 \rangle &= R \langle \Delta Y_-(\ell, t - \ell/c)^2 \rangle e^{-2q\ell} + T \langle \Delta Y_+^{vac}(0, t)^2 \rangle \\ &= e^{-2q\ell} R [R' \langle \Delta Y_+(\ell, t - \ell/c)^2 \rangle + T' \langle \Delta Y_-^{vac}(\ell, t - \ell/c)^2 \rangle] \\ &\quad + T \langle \Delta Y_+^{vac}(0, t)^2 \rangle \\ &= RR' e^{-4q\ell} \langle \Delta Y_+(0, t - 2\ell/c)^2 \rangle + T \langle \Delta Y_+^{vac}(0, t)^2 \rangle \\ &\quad + RT' e^{-2q\ell} \langle \Delta Y_-^{vac}(\ell, t - \ell/c)^2 \rangle. \end{aligned} \quad (14)$$

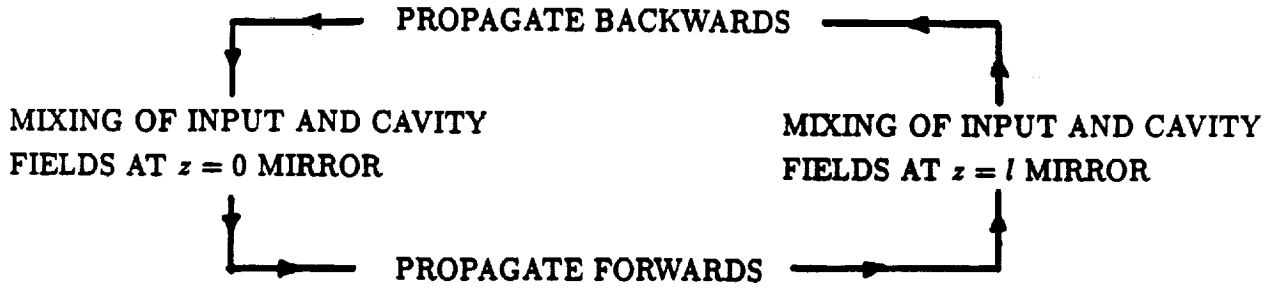


Fig. 3. Round Trip Evolution of Fields and Their Variances.

The foregoing sequence of mathematical steps in arriving at the round trip propagation of variances is shown diagrammatically in Fig. 3 to bring out the underlying physical picture.

In steady state, the quantum statistical properties of the field do not change from one round trip to the next. In this long-time limit, suppressing the time entry of each variance in Eq. (14), we get

$$\langle \Delta Y_+(0)^2 \rangle = \frac{[T \langle \Delta Y_+^{vac}(0)^2 \rangle + RT'e^{-2q\ell} \langle \Delta Y_-^{vac}(\ell)^2 \rangle]}{(1 - RR'e^{-4q\ell})}, \quad (15)$$

a result that is uniformly valid for all values of (R, T) and (R', T') pairs (with the obvious energy-conservation constraints, $R + T = R' + T' = 1$). It is also worth noting that in the derivation of (15), the only property of the input fields used was their white-noise (δ -correlated) character. Thus, (15) applies not just to vacuum-field inputs, but to arbitrary white-noise input fields.

In the good-cavity limit, $R, R' \simeq 1, q\ell \simeq 0$, we recover the result of Collett and Gardiner generalized to allow for arbitrary white-noise input fields at the two mirrors:

$$\langle \Delta Y_+(0)^2 \rangle \simeq \frac{T \langle \Delta Y_+^{vac}(0)^2 \rangle + T' \langle \Delta Y_-^{vac}(\ell)^2 \rangle}{(T + T') + 4q\ell}. \quad (16)$$

For vacuum-field inputs as explicitly indicated in Eq. (15), since the two input fields are statistically identical (except for their direction of propagation), we may write more simply

$$\langle \Delta Y_+(0)^2 \rangle = \frac{(T + RT'e^{-2q\ell})}{(1 - RR'e^{-4q\ell})} N_{vac}, \quad (17)$$

where

$$N_{vac} \equiv \langle \Delta Y_+^{vac}(0)^2 \rangle = \langle \Delta Y_-^{vac}(0)^2 \rangle.$$

Note that the calculation of the variance $\langle \Delta X_+(0)^2 \rangle$ of the X -quadrature of the intracavity is entirely analogous and is given by Eq. (17) provided q is replaced by $-q$ everywhere.

The degree of quadrature squeezing is the ratio $\langle \Delta Y_+(0)^2 \rangle / N_{\text{vac}}$ which is generally the factor by which two input fields with the same quadrature variance, but not necessarily vacuum fields, get squeezed on entering the cavity. Detailed discussions of this quantity in both textual and graphical forms have been presented elsewhere, where its generalization to include arbitrary relative phase between the two traveling components of the monochromatic pump has also been derived [5,6].

Having discussed the intracavity field, we now present the noise characteristics of the output field. Like the former field, the latter field is strongly correlated with the input fields as well. However, unlike the former, the output field quadratures can be easily subjected to a spectral analysis by choosing a sufficiently narrowband local oscillator field and integrating long enough in a balanced homodyne setup as was done in the original experiments [8]. We shall see that it is in this spectral sense that the output field exhibits a very high degree of squeezing.

The boundary connection of the output is similar to Eqs. (6). For example, the leftward-traveling output field at the $z = 0$ mirror is a linear superposition of the transmitted part of $e_-(0, t)$ and the reflected part of $e_+^{\text{vac}}(0, t)$. So any quadrature of the output field, say its Y_- -quadrature, obeys the boundary connection formula

$$Y_{\text{out}}(0, t) = \tilde{t}Y_-(0, t) + \tilde{r}Y_+^{\text{vac}}(0, t). \quad (18)$$

However, unlike the intracavity field, we must know the full time dependence of $Y_{\text{out}}(0, t)$, not just of its variance, before it can be spectrally analyzed. Equivalently, as (18) shows, we must know how $Y_-(0, t)$ evolves in time. But, that is easy to write down over a complete round trip since we know via Eqs. (9) and (11) how the intracavity field e_{\pm} interacts with the active medium in a single pass through it, while Eqs. (6) tell us how the input fields e_{\pm}^{vac} leak into the cavity at the $z = 0$ and $z = \ell$ mirrors. The round trip evolution of $Y_-(0, t)$ turns out to be

$$Y_-(0, t) = \tilde{r}\tilde{r}'e^{-2q\ell}Y_-(0, t - 2\ell/c) - \tilde{t}\tilde{r}'e^{-2q\ell}Y_+^{\text{vac}}(0, t - 2\ell/c) + \tilde{t}'e^{-q\ell}Y_-^{\text{vac}}(\ell, t - \ell/c), \quad (19)$$

which could also have been written down directly based on physical arguments presented below.

If $Y_-(0, t - 2\ell/c)$ is the Y -quadrature of the cavity field just before it is incident on the $z = 0$ mirror from the right then after that mirror reflection a fraction \tilde{r} of it is reflected while a fraction \tilde{t} of the input field $Y_+^{\text{vac}}(0, t - 2\ell/c)$ is transmitted. The two waves propagate rightward through the medium with their Y -quadratures attenuated by factor $e^{-q\ell}$. They are then reflected at the mirror at $z = \ell$ by factor $-\tilde{r}'$ while a fraction \tilde{t}' of the second input $Y_-^{\text{vac}}(0, t - \ell/c)$ is added to the circulating wave. The net field then propagates a distance ℓ leftward through the active medium, with its Y -quadrature attenuated further by $e^{-q\ell}$ as a result, to become the net field, given by the left-hand side of Eq. (19), a time ℓ/c later.

A Fourier analysis of Eq. (19) is straightforward. We shall focus only on the central (zero-detuning) frequency component since it has the largest noise reduction. Denoting the Fourier transform of a function $f(t)$ by $\tilde{f}(\delta\omega)$, we see that for $\delta\omega = 0$, Eq. (19) yields

$$\tilde{Y}_-(0, 0) [1 - \tilde{r}\tilde{r}'e^{-2q\ell}] = -\tilde{t}\tilde{r}'e^{-2q\ell}\tilde{Y}_+^{\text{vac}}(0, 0) + \tilde{t}'e^{-q\ell}\tilde{Y}_-^{\text{vac}}(\ell, 0),$$

while Eq. (18) yields

$$\tilde{Y}_{out}(0,0) = \tilde{i} \tilde{Y}_-(0,0) + \tilde{r} \tilde{Y}_+^{vac}(0,0).$$

By eliminating $\tilde{Y}_-(0,0)$ between these two relations and using the energy-conservation relation $\tilde{r}^2 + \tilde{i}^2 = 1$, one may easily show that

$$\tilde{Y}_{out}(0,0) = \frac{(\tilde{r} - \tilde{r}'e^{-2q\ell}) \tilde{Y}_+^{vac}(0,0) + \tilde{i}\tilde{i}'e^{-q\ell} \tilde{Y}_-^{vac}(\ell,0)}{(1 - \tilde{r}\tilde{r}'e^{-2q\ell})}, \quad (20)$$

whose variance is related to the spectral variance of (uncorrelated) input-field quadratures. If we assume that the input fields have the same spectral quadrature variance at a given frequency, such as is surely true for vacuum-field inputs then the spectral squeezing of the output field at zero detuning is by the factor

$$S_{out}^{(Y)}(0) \equiv \langle \Delta \tilde{Y}_{out}(0,0)^2 \rangle / \langle \Delta \tilde{Y}_+^{vac}(0,0)^2 \rangle = \frac{(\tilde{r} - \tilde{r}'e^{-2q\ell})^2 + \tilde{i}^2 \tilde{i}'^2 e^{-2q\ell}}{(1 - \tilde{r}\tilde{r}'e^{-2q\ell})^2}. \quad (21)$$

Just as for the cavity field, the ratio $\langle \Delta \tilde{X}_{out}(0,0)^2 \rangle / \langle \Delta \tilde{X}_+^{vac}(0,0)^2 \rangle$ for the X -quadrature is given by replacing q by $-q$ everywhere in relation (21).

It is worth noting that just below threshold $\tilde{r}\tilde{r}'e^{2q\ell} \rightarrow 1$, the X -quadrature of the output field at the $z = 0$ mirror has infinite variance in its central frequency component, while the corresponding Y -quadrature spectral component has a finite variance that depends on how large the transmission T' of the other mirror is. In particular, for $T' = 0$ regardless of the value of R (or of T), the output Y -quadrature has zero spectral variance at the line center. This is a very surprising result, implying as it does that even in a very low- Q but single-ended cavity the output field is perfectly squeezed in the spectral sense, if the parametric gain is high enough to drive the oscillator to its oscillation threshold. A more complete discussion of the output field, including the bandwidth of the squeezing spectrum, may be found in Ref. [6].

5 Squeezing in the Presence of Pump Noise

In a real experiment, pump noise is inevitable. Typically, the pump field has both amplitude and phase noise that can be described well in classical terms alone. For example, the pump amplitude may have a small fluctuating piece, described in Eq. (5) via a time dependent q ,

$$q(t) = q_0 + \delta q(t), \quad (22)$$

in which $\delta q(t)$ is an Ornstein-Uhlenbeck Gaussian process with zero mean and an exponentially decaying two-time correlation

$$\langle \delta q(t) \delta q(t') \rangle = \alpha_0 \Gamma e^{-\Gamma|t-t'|}. \quad (23)$$

The pump phase noise, on the other hand, is ultimately limited by phase diffusion which is well described by a classical Wiener-Levy Gaussian random process with zero mean value for the time derivative of the diffusing phase, $\delta\psi(t)$, and its two-time correlations:

$$\langle \delta \dot{\psi} \rangle = 0; \quad \langle \delta \dot{\psi}(t) \delta \dot{\psi}(t') \rangle = 2D\delta(t - t'). \quad (24)$$

The constants 2Γ and $2D$ are the amplitude and phase-noise contributions to the total pump linewidth.

Since detailed discussions of this problem have been presented elsewhere [6], we shall restrict our derivations here to its relatively simple but physically revealing aspects. To begin with, we shall take the white-noise limit, $\Gamma \rightarrow \infty$, for the amplitude noise. In more precise terms, this is the limit in which $\Gamma\ell/c \gg 1$.

Since q in Eqs. (5) and (8) is time dependent, the exponentials in Eqs. (9) and (11) have integrals in their exponents. For example, in Eq. (9b) one must replace

$$e^{-qz} \rightarrow e^{-q_0 z - \int_0^z \delta q(t-x'/c) dx'}$$

for a given statistical realization of δq . This means that the Y -quadrature variance is down by the factor

$$e^{-2q_0 \ell} \left\langle e^{-2 \int_0^\ell \delta q(t-x/c) dx} \right\rangle = e^{-2q_0 \ell + 4\alpha_0 \ell c}$$

in every single pass either leftward or rightward between the two mirrors. We used the familiar result that for a Gaussian random variable x ,

$$\langle e^x \rangle = e^{\langle x \rangle} e^{\frac{1}{2} \langle \Delta x^2 \rangle} \quad (25)$$

and the fact that when $\Gamma\ell/c \gg 1$,

$$\langle \delta q(t) \delta q(t') \rangle \simeq 2\alpha_0 \delta(t - t'), \quad (26)$$

to obtain the preceding factor.

A recognition of the extra factor $e^{4\alpha_0 \ell c}$ by which the Y -quadrature variance is altered when the pump amplitude has a fluctuating piece immediately tells us that Eqs. (15) and (17) must also be altered accordingly. Thus, for example, Eq. (17) takes the form

$$\langle \Delta Y_+(0)^2 \rangle = \frac{(T + RT' e^{-2q_0 \ell + 4\alpha_0 \ell c})}{(1 - RR' e^{-4q_0 \ell + 4\alpha_0 \ell c})} N_{vac}.$$

Since $\alpha_0 > 0$, the net effect of the δ -correlated pump amplitude fluctuations is to merely reduce the parametric attenuation of Y -quadrature fluctuations thereby leading to a smaller intracavity squeezing.

Although we have not discussed the opposite, static pump amplitude noise limit, $\Gamma\ell/c \ll 1$, it can be seen by physical arguments that for a given amplitude noise $\langle \delta q^2 \rangle^{\frac{1}{2}}$, the static case compromises intracavity squeezing more dramatically than the white-noise case, for it is roughly the zero-frequency Fourier component of the pump noise spectrum that controls the steady state characteristics of the signal field. As the noise bandwidth Γ increases, a fixed amount of amplitude noise is partitioned into more and more Fourier components, so that the zero-frequency component (like any other) goes down.

We turn now to the computation of spectral squeezing of the output field in the presence of a δ -correlated pump amplitude noise. This task is quite involved when compared with the derivation of the preceding intracavity variance formula. One must begin with the fluctuating analog of (19) which may be shown to be

$$Y_-(0, t) = \tilde{r}\tilde{r}'e^{-\eta(t)-\eta(t-\ell/c)}Y_-(0, t-2\ell/c) - \tilde{t}\tilde{r}'e^{-\eta(t)-\eta(t-\ell/c)}Y_+^{vac}(0, t-2\ell/c) + \tilde{t}'e^{-\eta(t)}Y_-^{vac}(\ell, t-\ell/c), \quad (27)$$

in which

$$\eta(t) \equiv \int_0^t [q_0 + \delta q(t-z/c)] dz. \quad (28)$$

A direct Fourier transform of Eq. (27) is not possible. We must compute first the two-time correlation functions $\langle Y_-(0, t)Y_-(0, t') \rangle$, $\langle Y_-(0, t)Y_+^{vac}(0, t') \rangle$, and $\langle Y_+^{vac}(0, t)Y_+^{vac}(0, t') \rangle$ that enter the output autocorrelation function $\langle Y_{out}(0, t)Y_{out}(0, t') \rangle$ via Eq. (18). A Fourier transform of the output correlation then furnishes the spectral variance. To compute the former two correlation functions, we solve Eq. (27) for $Y_-(0, t)$ iteratively in terms of Y_{\pm}^{vac} at successively earlier times, one differing from the next by the roundtrip time $2\ell/c$:

$$Y_-(0, t) = -\tilde{t}\tilde{r}' \sum_{n=0}^{\infty} (\tilde{r}\tilde{r}')^n e^{-\eta_{2n+2}(t)} Y_+^{vac}(0, t-2\ell(n+1)/c) + \tilde{t}' \sum_{n=0}^{\infty} (\tilde{r}\tilde{r}')^n e^{-\eta_{2n+1}(t)} Y_-^{vac}(\ell, t-\ell(2n+1)/c) \quad (29)$$

in which

$$\eta_p(t) \equiv \int_0^{p\ell} [q_0 + \delta q(t-z/c)] dz. \quad (30)$$

We may use the identity (25) and the white-noise approximation (26) to obtain the useful formula

$$\begin{aligned} \langle e^{-\eta_p(t)} e^{-\eta_{p'}(t)} \rangle &= e^{-(p+p')q_0\ell} e^{\frac{1}{2}[\langle \delta\eta_p^2 \rangle + \langle \delta\eta_{p'}^2 \rangle + 2\langle \delta\eta_p \delta\eta_{p'} \rangle]} \\ &= e^{-(p+p')q_0\ell} e^{\alpha_0 c \ell (p+p'+2p^<)}, \end{aligned} \quad (31)$$

in which $p^<$ is the smaller of (p, p') .

When combined with the δ -correlated nature of the vacuum fields, relation (29) enables one to secure the needed correlations from which the following output quadrature autocorrelation function is obtained [6]:

$$\begin{aligned} \langle Y_{out}(0, t) Y_{out}(0, t') \rangle &= \frac{C}{\pi^2} \left[\tilde{t}^2 + \tilde{r}^2 \tilde{t}'^2 e^{-2(q_0-2\alpha_0 c)\ell} \right. \\ &+ \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} (\tilde{r}\tilde{r}')^{n+n'} \delta(t-t'-2(n-n')\ell/c) e^{-2(n+n')q_0\ell} e^{2\alpha_0 c \ell (n^>+3n^<)} \\ &- \tilde{t}'^2 \sum_{n=0}^{\infty} (\tilde{r}\tilde{r}')^n [\delta(t-t'-2n\ell/c) + \delta(t-t'+2n\ell/c)] e^{-2nq_0\ell+2\alpha_0 n\ell} + \delta(t-t') \left. \right] \end{aligned} \quad (32)$$

in which

$$\langle Y_+^{vac}(0, t) Y_+^{vac}(0, t') \rangle = \langle Y_-^{vac}(\ell, t) Y_-^{vac}(\ell, t') \rangle \equiv C\delta(t-t').$$

As before, we are only interested here in the central frequency component of the quadrature spectrum. This is obtained from Eq. (32) by integrating it over $(t - t')$ in the range $(-\infty, \infty)$, which is a trivial task due to the presence of a δ -function in every term. The resulting infinite sums are related to the geometric series and can be carried out in closed form. The net result of these straightforward steps is the following noise reduction factor at line center:

$$S_{out}^{(Y)}(0) = \frac{1}{\tilde{r}^2} \left[1 + \frac{\tilde{t}^2 (\tilde{t}^2 + \tilde{r}^2 \tilde{t}'^2 e^{-2(\eta_0 - 2\alpha_0 c)\ell}) (1 + \tilde{r} \tilde{r}' e^{-2(\eta_0 - \alpha_0 c)\ell})}{(1 - \tilde{r}^2 \tilde{r}'^2 e^{-4(\eta_0 - 2\alpha_0 c)\ell}) (1 - \tilde{r} \tilde{r}' e^{-2(\eta_0 - \alpha_0 c)\ell})} - \frac{2\tilde{t}^2}{1 - \tilde{r} \tilde{r}' e^{-2(\eta_0 - \alpha_0 c)\ell}} \right]. \quad (33)$$

When the pump fluctuations are absent ($\alpha_0 = 0$), this expression naturally reduces to result (21). In general, however, a graphical presentation of (33) is imperative for physical insights. This is done in Fig. 4 for a symmetric cavity ($R = R'$). It is no surprise that as the pump amplitude fluctuations increase in strength, the amount of squeezing reduces for any fixed value of R (i.e. along a vertical line on the figure). For a fixed fluctuation strength, on the other hand, the higher its value the slower the squeezing increases, with increasing R , to its maximum value at oscillation threshold.

A reduction of the amplitude-noise bandwidth, so that $\Gamma\ell/c$ is no longer large compared to 1, leads to reduced output squeezing for the same reasons as for the intracavity field. It is worth noting that amplitude noise, being essentially multiplicative in nature (see Eq. (5)), is less important than pump phase noise which unavoidably couples the squeezed quadrature to the highly fluctuating quadrature, thereby seriously undermining squeezing.

6 Pump Phase Fluctuations

Even the quietest pump, such as one generated by a highly stable laser, has intrinsic random phase diffusion arising from the purely quantum mechanical process of spontaneous emission. This means that squeezing in the sub-harmonic signal field when measured relative to a fixed (or independently fluctuating) phase will show a time-dependent behavior as both the squeezed and unsqueezed orthogonal quadratures with phases slaved to the pump mix. However, if both the local oscillator (LO) and pump are derived from the same laser, then the reference LO phase and the phase of the ideally squeezed quadrature track each other. In spite of this phase tracking, there is a residual effect on squeezing, due to the time dependence of the pump phase diffusion [9], which we consider here.

In the presence of a finite $\delta\psi(t)$, as described by a Wiener-Levy Gaussian random process with moments (24), Eq. (5) has q replaced by $qe^{i\delta\psi(t)}$, and the signal quadratures $X_{\pm}(z, t)$ and $Y_{\pm}(z, t)$ are defined relative to the phase $\delta\psi(t)/2$:

$$\begin{aligned} X_{\pm}(z, t) &= \frac{1}{2} \left[e_{+}(z, t) e^{-i\delta\psi(t)/2} + e_{-}(z, t) e^{i\delta\psi(t)/2} \right]; \\ Y_{\pm}(z, t) &= \frac{1}{2i} \left[e_{+}(z, t) e^{-i\delta\psi(t)/2} - e_{-}(z, t) e^{i\delta\psi(t)/2} \right]. \end{aligned} \quad (34)$$

These quadratures evolve according to the matrix equation

$$\left(\frac{\partial}{\partial z} \pm \frac{1}{c} \frac{\partial}{\partial t} \right) V_{\pm}(z, t) = \left[q\sigma_3 \pm \frac{i}{2c} \delta\psi(t)\sigma_2 \right] V_{\pm}(z, t), \quad (35)$$

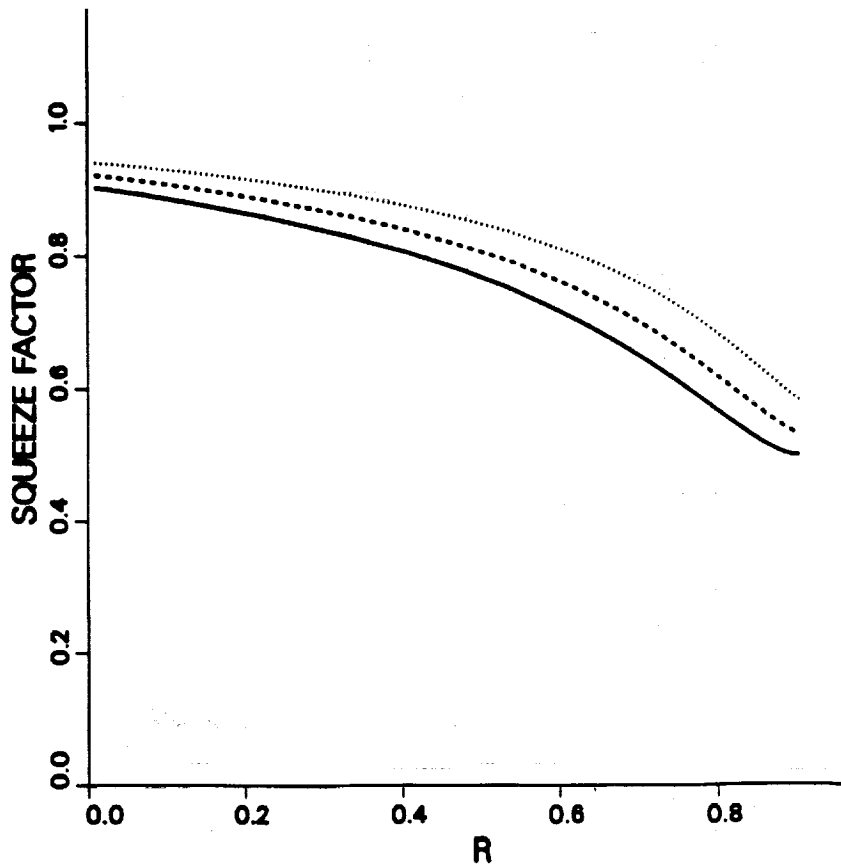


Fig. 4. Squeezing of the Central Frequency Component of the Output Field Quadrature in a Symmetric Cavity. The full, dashed, and dotted curves represent values of the fluctuation parameter $\alpha_0 cl$ equal to 0, 0.005, and 0.01, respectively, while the roundtrip gain coefficient $q_0 l$ is 0.05 in each case.

in which the column vector $V_{\pm}(z, t)$ is $(X_{\pm}(z, t), Y_{\pm}(z, t))^T$ and the σ 's are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (36)$$

Although Eq. (35) is a first order equation, it is a matrix equation with the coefficient matrix on the RHS at any time not committing with itself at another time. This renders the solution a formal one in terms of time-ordered or path-ordered exponentials. The path-ordering (or time-ordering) has however the advantage that successive path-ordered (or time ordered) exponentials from one roundtrip to the next may be easily multiplied. One first combines the solution of Eq. (35) with the boundary connections (9) to determine the single roundtrip evolution of $V_+(0, t)$ to obtain a matrix analog of Eq. (19). Iterative processing of such equation leads to a formal solution that can, via the simplicity of writing products of time (or path) ordered exponentials with contiguous limits as single time (path) ordered exponentials over the entire time (or path) interval, be expressed in the form

$$V_+(0, t) = \sum_{n=0}^{\infty} (\tilde{r}\tilde{r}')^n C(0, 2\ell, n; t) e^{2q\ell n\sigma_3} W^{vac}(t - 2n\ell/c). \quad (37)$$

In Eq. (37), W^{vac} is a column vector related to the quadratures of the two known input fields and $C(0, 2\ell n; t)$, a path-ordered matrix exponential involving an integral over $\delta\psi(t)$, represents the residual effect of pump phase diffusion over signal noise.

In Ref. [6], solution (37) serves as the starting point for computing the various variances and correlations needed for determining the steady-state intracavity quadrature variances and output-field quadrature noise spectrum. Eq. (37) is sufficiently complex that a statistical averaging over the phase noise $\delta\psi$, in spite of its Gaussian and δ -correlated nature, cannot be exactly performed in the involved integrals. One must settle for a series expansion of intracavity and output squeezing in powers of the phase diffusion constant D , which has been determined to $O(D^2)$ [6]. We refer the interested reader to that reference for more details. It suffices here to state that pump phase diffusion seems to be most important near threshold where the fluctuations in the X -quadrature of the cavity field have a highly slowed relaxation rate.

7 Conclusions

We have presented here an analysis of squeezing in a degenerate parametric oscillator that lends itself to an easy physical interpretation for the most part. For completeness, we have also summarized the impact of pump amplitude and phase noises of sorts encountered in a real experiment on the observed degrees of cavity and output squeezing. An exact analysis for the case of a finite pump-phase diffusion noise $\delta\psi(t)$ is beset by the difficulties of computing the statistical averages of path-ordered integrals involving $\delta\psi(t)$.

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